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WORMHOLES WITH A NUT CHARGE IN HIGHER CURVATURE THEORIES

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Abstract: We consider scalarized wormholes with a Newman-Unti-Tamburino charge in both Einstein-scalar-Gauss-Bonnet and Einstein-scalar-Chern-Simons theories. By varying the coupling parameter and the scalar charge we determine the scalarized wormhole solutions, and their dependence on the Newman-Unti-Tamburino charge. We generalize these scalarized wormhole solutions in two ways. On the one hand, we include a Newman-Unti-Tamburino charge and on the other hand we consider besides the Gauss-Bonnet invariant also the Chern-Simons invariant.

Keywords: wormholes, scalar fields, NUT charge, black holes, higher curvature theories

Introduction. An alternative higher curvature invariant to study curvature-induced spontaneously scalarized black holes is the Chern–Simons invariant. However, in the static case of the Schwarzschild metric the invariant vanishes, and therefore no spontaneously scalarized Schwarzschild black holes arise. This is different for the Kerr metric, since rotation leads to a finite Chern–Simons source term for the scalar field[1], which should allow for spontaneously scalarized Kerr black holes. In order to learn about scalarized rotating Chern–Simons black holes without having to deal with the full complexity of the challenging set of the resulting coupled partial differential equations, one may first resort to the technically much simpler case and include a Newman-Unti-Tamburino (NUT) charge instead of rotation, as pursued successfully by Brihaye et al.[2,3].

To obtain traversable wormhole solutions the energy conditions must be violated. In General Relativity this can be achieved by the presence of exotic matter. However, by allowing for alternative theories of gravity traversable wormholes can be obtained without the need for exotic matter[4]. Employing the string theory motivated dilaton-Gauss-Bonnet coupling, static scalarized wormholes were shown to exist, and their domain of existence and their properties were studied before[5-7]. In the dilatonic case, the black hole boundary of the domain of existence corresponds to static dilatonic black holes[8]. For other coupling functions, which give rise to spontaneously scalarized Gauss-Bonnet black holes, the black hole boundary of the domain of existence of scalarized wormholes consists of the corresponding spontaneously scalarized black holes [9].

Here we present the actions involving the Gauss-Bonnet and the Chern–Simons term and exhibit the equations of motion for both cases. We then discuss the boundary conditions, the conditions for the center, i.e., the throat (or equator), and the null energy condition. Subsequently we address the numerical procedure and present our results.

Action and equations of motion

We use the effective action for Einstein-scalar-higher curvature invariant theories in the following

$$S=\frac{1}{16\pi}\int\left[R-\frac{1}{2}\partial_{\mu}\phi\partial^{\mu}\phi+F(\phi)I(g)\right]\sqrt{-g}d^{4}x\,, \tag{1}$$

where R is the curvature scalar, and ϕ denotes the massless scalar field without self-interaction[10]. The scalar field is coupled with some coupling function $F(\phi)$ to an invariant $I(g)$. We choose the coupling function $F(\phi)$,

$$F(\phi)=\alpha\phi^{2}\,, \tag{2}$$

where α is the coupling constant, the simplest choice leading to curvature-induced spontaneous scalarization of black holes.

For the invariant $I(g)$ we make two choices, (i) the Gauss-Bonnet term

$$I(g)=R_{\rm GB}^{2}=R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}-4R_{\mu\nu}R^{\mu\nu}+R^{2}\,, \tag{3}$$

and (ii) the Chern-Simons term

$$I(g)=R_{\rm CS}^{2}={}^{*}R^{\mu}{}_{\nu}{}^{\rho\sigma}R^{\nu}{}_{\mu\rho\sigma}\,, \tag{4}$$

where the Hodge dual of the Riemann-tensor ${}^{*}R^{\mu}{}_{\nu}{}^{\rho\sigma}=\frac{1}{2}\eta^{\rho\sigma\kappa\lambda}R^{\mu}{}_{\nu\kappa\lambda}$ is defined with the 4-dimensional Levi-Civita tensor $\eta^{\rho\sigma\kappa\lambda}=m^{\rho\gamma\sigma\tau}/\sqrt{-g}$. While both invariants are topological in four dimensions, the coupling to the scalar field ϕ via the coupling function $F(\phi)$ provides significant contributions to the equations of motion.

We obtain the coupled set of field equations by varying the action (1) with respect to the scalar field and to the metric,

$$\nabla^{\mu}\nabla_{\mu}\phi+\frac{dF(\phi)}{d\phi}I=0\,, \tag{5}$$

$$G_{\mu\nu}=\frac{1}{2}T_{\mu\nu}^{(\rm eff)}\,, \tag{6}$$

where $G_{\mu\nu}$ is the Einstein tensor and $T_{\mu\nu}^{(\rm eff)}$ denotes the effective stress-energy tensor

$$T_{\mu\nu}^{(\rm eff)}=T_{\mu\nu}^{(\phi)}+T_{\mu\nu}^{(I)}\,, \tag{7}$$

which consists of the scalar field contribution

$$T_{\mu\nu}^{(\phi)} = \left(\nabla_\mu \phi\right)\left(\nabla_\nu \phi\right) - \frac{1}{2} g_{\mu\nu}\left(\nabla_\rho \phi\right)\left(\nabla^\rho \phi\right), \tag{8}$$

and a contribution from the respective invariant $I(g)$. For the chosen invariants we obtain (i)

$$T_{\mu\nu}^{(GB)} = \left(g_{\rho\mu} g_{\lambda\nu} + g_{\lambda\mu} g_{\rho\nu}\right) \eta^{\kappa\lambda\alpha\beta *} R^{\rho\gamma}{}_{\alpha\beta} \nabla_\gamma \nabla_\kappa F(\phi), \tag{9}$$

and (ii)

$$T^{(CS)\mu\nu} = -8\left[\nabla_\rho F(\phi)\right] \mathrm{T}^{\rho\sigma\tau\mu} (\nabla_\tau R^\nu)_\sigma + \left[\nabla_\rho \nabla_\sigma F(\phi)\right]^* R^{\sigma(\mu\nu)\rho}. \tag{10}$$

To obtain static, spherically symmetric wormhole solutions with a NUT charge N we assume the line element to be of the form

$$ds^2 = -e^{f_0}\left(dt - 2N\cos\theta d\varphi\right)^2 + e^{f_1}\left[dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right)\right]. \tag{11}$$

All three functions, the two metric functions f_0, f_1 and the scalar field function ϕ, depend only on the radial coordinate r.

When we insert the above line element (11) for the metric and the scalar field into the scalar-field equation (5) and the Einstein equations (6) with effective stress-energy tensor (7), we obtain five coupled, nonlinear ordinary differential equations. However, these are not independent, since the θ-dependence factorizes, and one ordinary differential equation can be treated as a constraint. This leaves us with three coupled ordinary differential equations of second order. Note that in case (i) the system can be reduced to one first order and two second order ordinary differential equations.

Throats, equators, and boundary conditions

In order to obtain scalarized nutty wormhole solutions, we need to impose an appropriate set of boundary conditions for the ordinary differential equations, which we now address. We first introduce the circumferential (or spherical) radius

$$R_C = e^{\frac{f_1}{2}} r, \tag{12}$$

of the wormhole spacetimes, which may possess one or more finite extrema. If there is a single finite extremum, this corresponds to the single throat of the respective wormholes. Here we will mainly consider such single throat wormholes, thus featuring a single minimum. But wormholes with more extrema may also exist. They might, for instance, possess a local maximum surrounded by two minima. The local maximum would then correspond to their equator, while the two minima would represent their two throats, making them double throat wormholes.

To obtain the first set of boundary conditions we therefore require the presence of an extremum of the spherical radius at some $r = r_0$. This yields

$$\left.\frac{dR_C}{dr}\right|_{r=r_0} = 0 \Leftrightarrow \left.\frac{df_1}{dr}\right|_{r=r_0} = -\frac{2}{r_0}. \tag{13}$$

Some details on the condition $R_C''(r_0) > 0$ are given in the Appendix. In the following we will refer to the two-dimensional submanifolds defined by $r = r_0$ and $t = const$. as the center of the configurations.

We obtain the second set of boundary conditions by requiring the usual boundary conditions for $r \to \infty$. The associated asymptotic expansions of the metric functions and the scalar field read

$$f_0 = -\frac{2M}{r} + O\left(r^{-3}\right), \tag{14}$$

$$f_1 = \frac{2M}{r} + O\left(r^{-2}\right), \tag{15}$$

$$\phi = \phi_\infty - \frac{D}{r} + O\left(r^{-3}\right), \tag{16}$$

where M denotes the mass of the wormholes and D corresponds to their scalar charge. The quantity ϕ_∞ represents the asymptotic value of the scalar field. We note that all higher order terms in the expansion can be expressed in terms of M, D and ϕ_∞. Thus the solution is uniquely determined by these quantities (and parameters of the theory).

Energy conditions

In wormhole solutions the null energy condition

$$T_{\mu\nu} n^\mu n^\nu \geq 0, \tag{17}$$

must be violated, where n^μ is any null vector ($n^\mu n_\mu = 0$). Thus it is sufficient to show that null vectors exist, such that $T_{\mu\nu} n^\mu n^\nu < 0$ in some spacetime region. Such a null vector n^μ is given by $n^\mu = \left(1, \sqrt{-g_{tt} / g_{\eta\eta}}, 0, 0\right)$, and thus $n_\mu = \left(g_{tt}, \sqrt{-g_{tt}\, g_{\eta\eta}}, 0, 0\right)$. The null energy condition then takes the form

$$T_{\mu\nu} n^\mu n^\nu = T_t^t n^t n_t + T_\eta^\eta n^\eta n_\eta = -g_{tt}\left(-T_t^t + T_\eta^\eta\right). \tag{18}$$

Consequently the null energy condition is violated when

$$-T_t^t + T_\eta^\eta < 0. \tag{19}$$

Alternatively, considering the null vector

$$n^\mu = \left(1, 0, \sqrt{-g_{tt} / g_{\theta\theta}}, 0\right), \tag{20}$$

the null energy condition is violated when

$$-T_t^t + T_\theta^\theta < 0. \tag{21}$$

These conditions have been addressed before for various scalarized wormhole solutions [9-11].

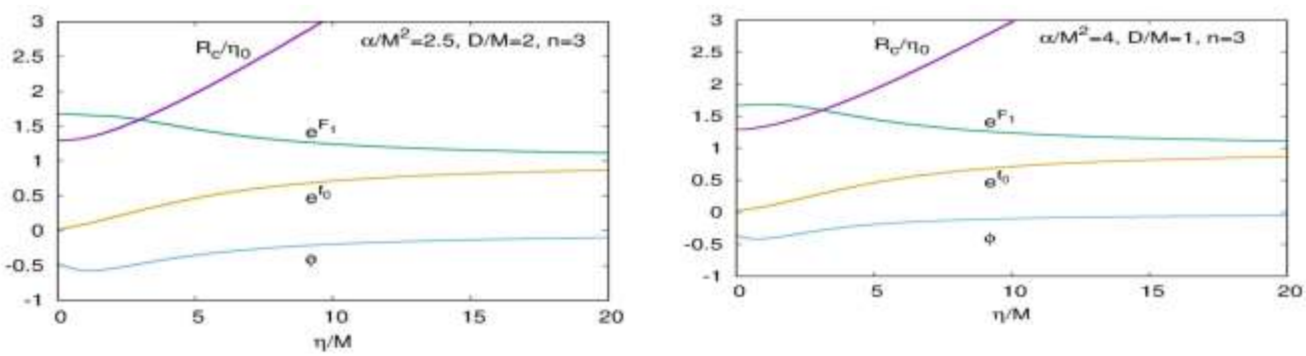
Results. In order to solve the coupled Einstein and scalar field equations numerically we introduce the inverse radial coordinate $x = 1/r$. The asymptotic region $r \to \infty$ then corresponds to $x \to 0$. In this region the expansion of the metric functions and the scalar field reads (see Equations. (14)-(16))

$$f_0 = -2M\,x + O\left(x^3\right), \quad f_1 = 2M\,x + O\left(x^2\right), \quad \phi = \phi_\infty - D\,x + O\left(x^3\right). \qquad (22)$$

We treat the system of ordinary differential equations as an initial value problem, for which we employ the fourth order Runge Kutta method. From the above expansion we read off the initial values,

$$f_{0,\text{ini}} = 0, \quad f'_{0,\text{ini}} = -2M\,, \quad f_{1,\text{ini}} = 0\,, \quad f'_{1,\text{ini}} = 2M\,, \quad \phi_{\text{ini}} = 0\,, \quad \phi'_{\text{ini}} = -D\,. \qquad (23)$$

The computation then starts at spatial infinity, $x = 0$, and ends at the center at some finite $x = x_0$, where the condition (13) is reached.



a) b)

Figure 1. Examples of nutty wormhole solutions: Metric profile functions e^{f_0}, e^{F_1}, scalar field function ϕ, and scaled circumferential radius R_c/η_0 vs radial coordinate η/M; (a) Gauss-Bonnet with parameters $\alpha/M^2 = 2.5$, $D/M = 2$ and $n = N/M = 3$. (b) Chern-Simons with $\alpha/M^2 = 4$, $D/M = 1$ and $n = N/M = 3$.

By following the above numerical procedure we obtain the sets of nutty wormhole solutions for both invariants $I(g)$. Here we demonstrate some typical solutions for both cases. We exhibit in Figure. 1 the metric profile functions e^{f_0}, e^{F_1}, and the scalar field function ϕ versus the radial coordinate η/M for the Gauss-Bonnet invariant (a) and the Chern–Simons invariant (b), choosing parameters $\alpha/M^2 = 2.5$, $D/M = 2$ and $n = N/M = 3$, and $\alpha/M^2 = 4$, $D/M = 1$ and $n = N/M = 3$, respectively. The figures also show the circumferential radius R_c/η_0 versus η/M. As required, R_c reaches an extremum at the center $\eta = 0$. We note that the solutions have rather similar properties for both invariants.

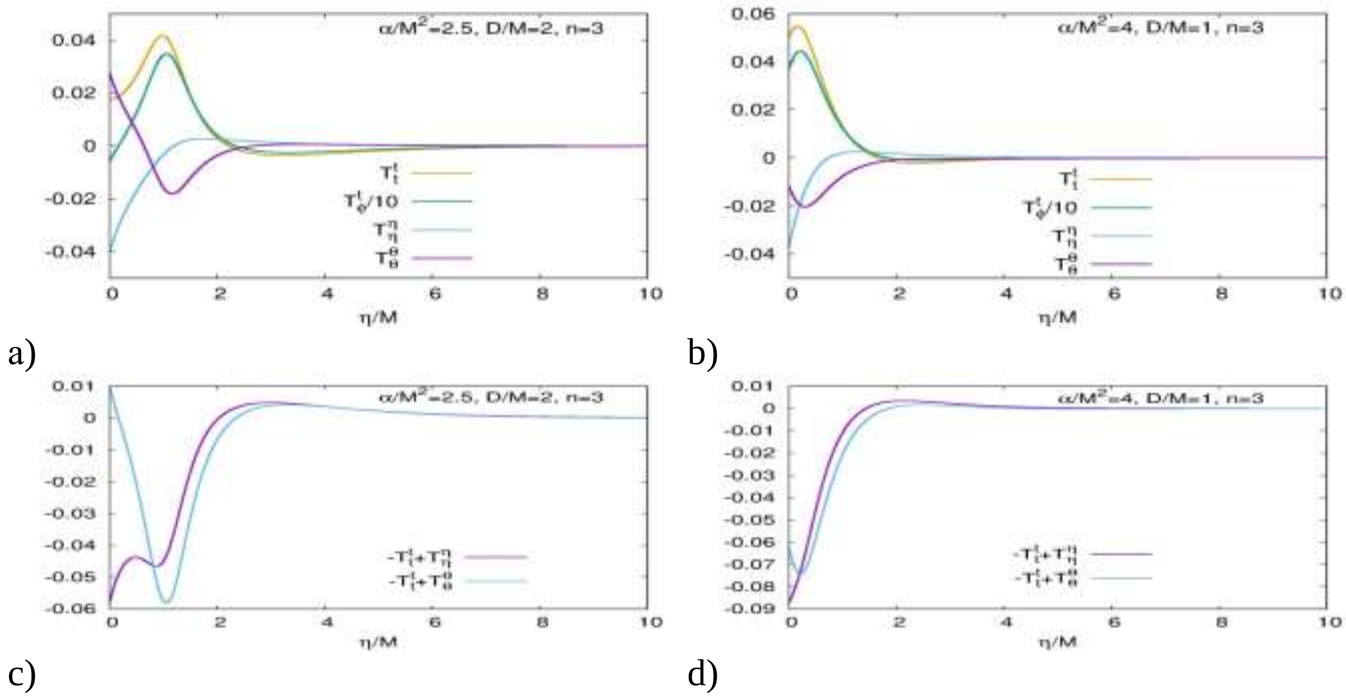


Figure 2.Examples of nutty wormhole solutions: (c) and (d) stress-energy tensor components T_t^t, T_ϕ^t, T_η^η, and T_θ^θ vs radial coordinate η / M ;(e) and (f) null energy condition conditions $-T_t^t + T_\eta^\eta \geq 0$ and $-T_t^t + T_\theta^\theta \geq 0$ vs radial coordinate η / M .

To see that the wormhole solutions violate the energy conditions, we inspect the components of the effective energy momentum tensor, T_t^t, T_ϕ^t, T_η^η, and T_θ^θ. These are shown for the same solutions and the Gauss-Bonnet and Chern–Simons invariants in Figures. 2(a) and 1(b), respectively. In particular, we note, that the component T_η^η is negative in the vicinity of the center for both invariants. Moreover, all components are negative in some region of the spacetime. We exhibit in Figures. 1(c) and 1(d) the null energy conditions $-T_t^t + T_\eta^\eta \geq 0$ and $-T_t^t + T_\theta^\theta \geq 0$ for the Gauss-Bonnet and Chern–Simons invariants, respectively. The figures clearly demonstrate the null energy condition violation for both invariants.

Conclusions. We have investigated scalarized wormhole solutionswith a NUT charge in these higher curvature theories. The presence of a NUT charge leads to solutions with a Misner string on the polar axis. However, the dependence of the polar angle factorizes and thus only a set of coupled ordinary differential equations for the metric functions and the scalar field arises. Moreover the usual boundary conditions at spatial infinity are retained.

Solving numerically the ordinary differential equations we have obtained scalarized wormhole solutions a NUT charge for both higher curvature invariants. These wormhole solutions possess a minimum of the circumferential radius, that arises naturally in these solutions, and that is identified with the wormhole throat. When integrating the ordinary differential equations beyond the throat, a maximum may be encountered, that would correspond to an equator.

The presence of a NUT charge can be viewed as model for learning about scalarized rotating wormhole solutions. Their construction still represents a challenging task, not only because complicated sets of coupled partial differential equations need to be solved, but also because the proper conditions for the throat need to be formulated together with the proper set of junction conditions.

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References

[1]. Konno K., Matsuyama T., Tanda S. Rotating black hole in extended Chern–Simons modified gravity. Prog. Theor. Phys. 2009, 122, 561.
[2]. Brihaye Y., Herdeiro C., Radu E. The scalarised Schwarzschild-NUT spacetime. Phys. Lett. B 2019, 788, 295.
[3]. Brihaye, Y.; Radu, E. Remarks on the Taub-NUT solution in Chern–Simons modified gravity. Phys. Lett. B 2017, 764, 300.
[4]. Hochberg D. Lorentzian wormholes in higher order gravity theories. Phys. Lett. 1990, B 251, 349.
[5]. Antoniou G., Bakopoulos A., Kanti P., Kleihaus B., Kunz, J. Novel Einstein-scalar–Gauss–Bonnet wormholes without exotic matter. Phys. Rev. D 2020, 101, 024033.
[6]. Kanti P., Kleihaus B., Kunz J. Wormholes in Dilatonic Einstein-Gauss–Bonnet Theory. Phys. Rev. Lett. 2011, 107, 271101.
[7]. Kanti P., Kleihaus B., Kunz J. Stable Lorentzian Wormholes in Dilatonic Einstein-Gauss–Bonnet Theory. Phys. Rev. D 2012, 85, 044007.
[8]. Kanti P., Mavromatos, N.E., Rizos J., Tamvakis K., Winstanley E. Dilatonic Black Holes in Higher Curvature String Gravity. Phys. Rev. D 1996, 54, 5049.
[9]. Ibadov R., Kleihaus B., Kunz J., Murodov S. Wormholes in Einstein-scalar–Gauss–Bonnet theories with a scalar self-interaction potential. Phys. Rev. D 2020, 102, 064010.
[10]. Ibadov R., Kleihaus B., Kunz J., Murodov S. Scalarized Nutty Wormholes. Symmetry 2021, 13, 89.
[11]. Morris M.S., Thorne K.S. Wormholes in space-time and their use for interstellar travel: A tool for teaching general relativity. Am. J. Phys. 1988, 56, 395.